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THE FIRST PASSAGE TIME AND AN ESTIMATE OF THE NUMBER OF LEVEL-CROSSINGS FOR A TELEGRAPH PROCESS

ЧАС ПЕРШОГО ДОСЯГНЕННЯ ТА ОЦІНКА ЧИСЛА ПЕРЕТИНІВ РІВНЯ ДЛЯ ТЕЛЕГРАФНИХ ПРОЦЕСІВ

It is a well-known result that almost all sample paths of Brownian motion or Wiener process $\{W(t)\}$ have infinitely many zero-crossings in the interval $(0,\delta)$ for $\delta > 0$. Under the Kac condition, the telegraph process weakly converges to the Wiener process. We estimate the number of intersections of a level or the number of level-crossings for the telegraph process. Passing to the limit under the Kac condition, we also obtain an estimate for the level-crossings for the Wiener process.

Відомо, що майже всі вибіркові траєкторії броунівського руху чи вінерівського процесу $\{W(t)\}$ мають нескінченно багато нульових перетинів в інтервалі $(0,\delta)$ при $\delta > 0$. За умови Каца телеграфний процес слабко збігається до вінерівського процесу. В роботі оцінюється число перетинів рівня для телеграфного процесу. Переходячи до границі за умови Каца, ми також отримуємо оцінку перетинів рівня для вінерівського процесу.

1. Introduction. Let us set the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. On the phase space $\mathbb{T} = \{0,1\}$ consider an alternating Markov stochastic process $\{\xi(t),\, t \geq t\}$, having the sojourn time τ_i corresponding to the state $x = i \in \mathbb{T}$, and generating matrix

$$Q = \lambda \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Denote by $\{x(t),\, t \geq 0\}$ the associated telegraph process. Then

$$\frac{d}{dt}x(t) = v(-1)^{\xi(t)}, \qquad v = \text{constant} > 0,$$

and $x(0) = x_0$.

2. Distribution of the first passage time. In this section we will find the explicit mathematical form for the distribution of the first passage time of a specific level $\mathbb{L}$ of a telegraph process on the real line.

Let assume a fixed level $\mathbb{L}$, then let us define $\Delta(t) = \mathbb{L} - x(t)$. Furthermore, we assume that $z = \mathbb{L} - x_0 > 0$.

Suppose $\xi(0) = k$ and define

$$T_k(z) = \inf\{t \geq 0 : \Delta(t) = 0\}, \quad k \in \{0,1\},$$

i.e., $T_k(z)$ is the first passage time of the level $\mathbb{L}$ by the telegraph process $\{x(t)\}$ after assuming $\xi(0) = k$.

Now, let us denote as $f_k(t,z)\,dt = P(T_k(z) \in dt)$ the probability density function (pdf) of $T_k(z)$.

Theorem 1. *For* $t \geq \dfrac{z}{v}$

$$f_0(t,z) = e^{-\lambda t}\left[\delta(z-vt) + \frac{\lambda z}{v^2}\,\frac{I_1\left(\dfrac{\lambda}{v}\sqrt{v^2t^2-z^2}\right)}{\sqrt{v^2t^2-z^2}}\right],$$

$$f_1(t,z) = e^{-\lambda t}\left[\frac{I_1\left(\lambda\left(t-\dfrac{z}{v}\right)\right)}{t-\dfrac{z}{v}} + \lambda z \int\limits_{z/v}^{t} \frac{I_1(\lambda(t-u))I_1\left(\dfrac{\lambda}{v}\sqrt{v^2u^2-z^2}\right)}{(t-u)\sqrt{v^2u^2-z^2}}\,du\right],$$

where $I_1(\cdot)$ *is the modified Bessel function of the first kind.*

Proof. Consider the Laplace transform of $T_k(z)$, $k \in \{0,1\}$,

$$\varphi_k(s,z) = \mathbf{E}\left[e^{-sT_k(z)}\right], \quad s \geq 0,$$

and by using renewal theory concepts we can obtain the following system of integral equations for these Laplace transforms, i.e.,

$$\varphi_0(s,z) = e^{-\frac{s+\lambda}{v}z} + \frac{\lambda}{v}\int\limits_0^z e^{-\frac{s+\lambda}{v}u}\varphi_1(s,z-u)\,du =$$

$$= e^{-\frac{s+\lambda}{v}z}\left[1 + \frac{\lambda}{v}\int\limits_0^z e^{\frac{s+\lambda}{v}u}\varphi_1(s,u)\,du\right],$$

$$\varphi_1(s,z) = \frac{\lambda}{v}e^{\frac{s+\lambda}{v}z}\int\limits_z^\infty e^{-\frac{s+\lambda}{v}u}\varphi_0(s,u)\,du.$$

We then differentiate these two equations to obtain the following system:

$$\frac{\partial}{\partial z}\varphi_0(s,z) = -\frac{s+\lambda}{v}\varphi_0(s,z) + \frac{\lambda}{v}\varphi_1(s,u),$$

$$\frac{\partial}{\partial z}\varphi_1(s,z) = \frac{s+\lambda}{v}\varphi_1(s,z) - \frac{\lambda}{v}\varphi_0(s,u).$$

It is well-known, Pogorui and Rodríguez-Dagnino [1], that this set of equations relating $\varphi_0(s,u)$ and $\varphi_1(s,u)$ can be represented as

$$Mf = 0,$$

where

$$M = \begin{bmatrix} \dfrac{\partial}{\partial z} + \dfrac{s+\lambda}{v} & -\dfrac{\lambda}{v} \\ \dfrac{\lambda}{v} & \dfrac{\partial}{\partial z} - \dfrac{s+\lambda}{v} \end{bmatrix}.$$

Then $f(z)$ satisfies the following equation:

$$(\mathrm{Det}(M))f = 0,$$

where $\mathrm{Det}(M)$ is the determinant of the matrix M.

By calculating the determinant, we have

$$\frac{\partial^2}{\partial z^2} f(z) - \frac{s^2 + 2\lambda s}{v^2} f(z) = 0.$$

The solution of this equation has the form

$$f(z) = C_1\, e^{\sqrt{s^2+2\lambda s}\frac{z}{v}} + C_2\, e^{-\sqrt{s^2+2\lambda s}\frac{z}{v}}.$$

The constants C_1 and C_2 are obtained from the conditions imposed on the system of integral equations, and we can obtain the solutions

$$\varphi_0(s,z) = e^{-\frac{z}{v}\sqrt{s(s+2\lambda)}} \tag{1}$$

and

$$\varphi_1(s,z) = \frac{s + \lambda - \sqrt{s(s+2\lambda)}}{\lambda}\, e^{-\frac{z}{v}\sqrt{s(s+2\lambda)}}. \tag{2}$$

The inverse Laplace transform of $\varphi_0(s,z)$ with respect to s yields the following (generalized) pdf (see [2, p. 239], formula 88):

$$f_0(t,z) = \mathcal{L}^{-1}\left(e^{-\frac{z}{v}\sqrt{s(s+2\lambda)}}, t\right) =$$

$$= e^{-\lambda t}\delta(z - vt) + z\lambda e^{-\lambda t}\frac{I_1\left(\dfrac{\lambda}{v}\sqrt{v^2t^2 - z^2}\right)}{\sqrt{v^2t^2 - z^2}}, \quad t \geq \frac{z}{v}. \tag{3}$$

Hence,

$$P(T_0(z) \in dt) = e^{-\lambda t}\delta(z - vt)\, dt + z\lambda e^{-\lambda t}\frac{I_1\left(\dfrac{\lambda}{v}\sqrt{v^2t^2 - z^2}\right)}{\sqrt{v^2t^2 - z^2}}\, dt, \quad t \geq \frac{z}{v}.$$

Similarly, the inverse Laplace transform of the first term of $\varphi_1(s,z)$ can be obtained from Bateman and Erdélyi [3, p. 237] and [4, p. 284] (formula (11))

$$\mathcal{L}^{-1}\left(\frac{s + \lambda - \sqrt{s(s+2\lambda)}}{\lambda}, t\right) = \frac{1}{\lambda}\mathcal{L}^{-1}\left((s + \lambda - \sqrt{s(s+2\lambda)}), t\right) = \frac{e^{-\lambda t}}{t} I_1(\lambda t). \tag{4}$$

The inverse Laplace transform of $\varphi_1(s,z)$ is just the convolution of Eqs. (3) and (4). Thus, the pdf $f_1(t,z)$ is given by

$$f_1(t,z) = \int\limits_{z/v}^{t} \frac{e^{-\lambda(t-u)}}{(t-u)} I_1(\lambda(t-u)) \left[e^{-\lambda u}\delta(z - vu) + z\lambda e^{-\lambda u}\frac{I_1\left(\dfrac{\lambda}{v}\sqrt{v^2u^2 - z^2}\right)}{\sqrt{v^2u^2 - z^2}}\right] du =$$

$$= \frac{e^{-\lambda t}}{t - \frac{z}{v}} I_1\left(\lambda\left(t - \frac{z}{v}\right)\right) + z\lambda e^{-\lambda t} \int\limits_{z/v}^{t} \frac{I_1(\lambda(t-u))\, I_1\left(\frac{\lambda}{v}\sqrt{v^2u^2 - z^2}\right)}{(t-u)\sqrt{v^2u^2 - z^2}}\, du$$

for $t \geq \frac{z}{v}$.

Theorem 1 is proved.

3. Estimation of the number of level-crossings for a telegraph process. We denote as $C_k(t, z)$ the number of intersections of level z made by the particle $x(t)$ during the time interval $(0, t)$, $t > 0$, assuming that $\xi(0) = k \in \{0, 1\}$. We consider the renewal function $H_k(t, z) = \mathbf{E}[C_k(t, z)]$.

Now, let us consider the so-called Kac's condition (or the hydrodynamic limit), i.e., let $\lambda = \varepsilon^{-2}$, $v = c\varepsilon^{-1}$, then as $\varepsilon \to 0$ (or $\lambda \to \infty$ and $v \to \infty$) we have that $\frac{v^2}{\lambda} \to c^2$.

It was proved in [5] that, under Kac's condition, the telegraph process $x(t)$ weakly converges to the Wiener process $W(t)$ which is normal distributed as $N(0, ct)$.

Theorem 2. *Under Kac's condition we have*

$$\lim_{\lambda\to\infty} = \frac{H_k(t,0)}{\sqrt{\lambda}} = \lim_{v\to\infty} \frac{c\, H_k(t,0)}{v} = \lim_{\varepsilon\to 0} \varepsilon c\, H_k(t,0) = c\sqrt{\frac{2}{\pi}}\,\sqrt{t}.$$

Proof. The Laplace transform of the general renewal function will be used in this proof, see the seminal book on this subject Cox [6]. It follows from (2) that the Laplace transform $\widehat{H}_1(s, 0) = \mathcal{L}(H_1(t, 0), t)$ of $H_1(t, 0)$ with respect to t has the form

$$\widehat{H}_1(s,0) = \frac{1}{s}\sum_{k=0}^{\infty}\left(\frac{s + \lambda - \sqrt{s(s+2\lambda)}}{\lambda}\right)^k = \frac{\lambda}{s\sqrt{s(s+2\lambda)} - s^2}.$$

It is not hard to verify that

$$\mathbf{E}[C_1(t,0)] = \mathcal{L}^{-1}\left(\frac{\lambda}{s\sqrt{s(s+2\lambda)} - s^2}\right) =$$

$$= \frac{1}{2} + \left(\left(\frac{1}{2} + \lambda t\right) I_0(2\lambda t) + \lambda t I_1(2\lambda t)\right) e^{-\lambda t}, \quad t \geq 0.$$

Hence,

$$\lim_{\lambda\to\infty} \frac{\mathbf{E}[C_1(t,0)]}{\sqrt{\lambda}} = \sqrt{\frac{2}{\pi}}\,\sqrt{t}.$$

Theorem 2 is proved.

Taking into account $\lambda = \varepsilon^{-2}$, $v = c\varepsilon^{-1}$, we have

$$H_1(t,0) = \sqrt{\lambda}\frac{2}{\pi}\,\sqrt{t} = c\sqrt{\frac{2}{\pi}}\,\sqrt{t}\, v$$

as $v \to \infty$.

Therefore, for a fixed $t > 0$ the number of crossings of level 0 by the telegraph process, under Kac's condition, goes to ∞ as the velocity v.

We should notice that the result of Theorem 2 is in correspondence with results of M. I. Portenko [7].

Now, let us denote as

$$F_1(x) = \int_0^x f_1(t,0)\,dt = \int_0^x \frac{e^{-\lambda t}}{t} I_1(\lambda t)\,dt.$$

Theorem 3.

$$\mathrm{P}\left\{\left(1-\int_0^{\lambda x}\frac{e^{-u}}{u}I_1(u)\,du\right)C_1(x,0)\geq\frac{3}{\sqrt{y}}\right\}\to G_{1/2}(y)$$

as $\lambda\to\infty$. The cumulative distribution function (cdf) $G_{1/2}(y)$ is the one-sided stable distribution satisfying the condition $y^{1/2}[1-G_{1/2}(y)]\to 3$ as $y\to\infty$ [8].

Proof. It is easily seen that

$$\lim_{x\to\infty}\sqrt{x}(1-F_1(x)) = \lim_{x\to\infty}\sqrt{x}\left(1-\int_0^{\lambda x}\frac{e^{-2\lambda t}}{t}I_1(2\lambda t)\,dt\right) =$$

$$= 2\lim_{x\to\infty} e^{-2\lambda x}x^{1/2}I_1(2\lambda x) = \frac{1}{\sqrt{\lambda\pi}}.$$

Therefore, $L(x)=\sqrt{x}(1-F_1(x))$ is slowly varying and

$$1-F_1(x) = x^{-1/2}L(x).$$

Since $F_1(x) = \int_0^x \frac{e^{-\lambda t}}{t} I_1(\lambda t)\,dt = \int_0^{\lambda x}\frac{e^{-u}}{u}I_1(u)\,du$ and $\lambda\to\infty$ we have $F_1(x) = F(s) = = s^{-1/2}L(s)$, where for a fixed $x>0$, $s=\lambda x\to\infty$. By using a result in [8, p. 373] (Chapter XI.5), we obtain

$$\mathrm{P}\left\{\left(1-\int_0^{\lambda x}\frac{e^{-u}}{u}I_1(u)\,du\right)C_1(x,0)\geq\frac{3}{\sqrt{y}}\right\}\to G_{1/2}(y)$$

as $\lambda\to\infty$, where the cdf $G_{1/2}(y)$ is the one-sided stable distribution satisfying the condition $y^{1/2}[1-G_{1/2}(y)]\to 3$ as $y\to\infty$.

Theorem 3 is proved.

Therefore, under Kac's condition the number of crossing of a level by the telegraph process, i.e., $C_1(x,0)$, is of the order of magnitude $\sqrt{\lambda}=v$.

4. Estimation of the number of level-crossings in higher dimensions. Firstly, let us consider the following modification of a telegraph process. Suppose $\{\theta_k,\ k\geq 1\}$ is a sequence of independent identically distributed exponential random variables with common rate $\lambda>0$ and $\tau_n=\sum_{k=1}^n\theta_k$, $n\geq 1$. A particle starts its motion on a line from the origin and moves in one of two directions with probability $1/2$ during the random time θ_1. At epoch τ_n, $n\geq 1$, the particle chooses one of two directions on the line with probability $1/2$, and keeps moving along this direction with velocity v. By using the notation stated in Section 1, for this process we have

$$\varphi_0(s,z) = e^{-\frac{s+\lambda}{v}z} + \frac{\lambda}{2v}\int\limits_0^z e^{-\frac{s+\lambda}{v}u}[\varphi_0(s,z-u)+\varphi_1(s,z-u)]\,du =$$

$$= e^{-\frac{s+\lambda}{v}z}\left[1 + \frac{\lambda}{2v}\int\limits_0^z e^{\frac{s+\lambda}{v}u}[\varphi_0(s,u)+\varphi_1(s,u)]\,du\right],$$

$$\varphi_1(s,z) = \frac{\lambda}{2v}e^{\frac{s+\lambda}{v}z}\int\limits_z^\infty e^{-\frac{s+\lambda}{v}u}[\varphi_0(s,u)+\varphi_1(s,u)]\,du.$$

We then differentiate these two equations to obtain the following system:

$$\frac{\partial}{\partial z}\varphi_0(s,z) = -\frac{s+\lambda/2}{v}\varphi_0(s,z) + \frac{\lambda}{2v}\varphi_1(s,u),$$

$$\frac{\partial}{\partial z}\varphi_1(s,z) = \frac{s+\lambda/2}{v}\varphi_1(s,z) - \frac{\lambda}{2v}\varphi_0(s,u).$$

Much in the same manner as we obtained Eqs. (1), (2)

$$\varphi_0(s,z) = e^{-\frac{z}{v}\sqrt{s(s+\lambda)}},$$

and

$$\varphi_1(s,z) = \frac{2s+\lambda-2\sqrt{s(s+\lambda)}}{\lambda}e^{-\frac{z}{v}\sqrt{s(s+\lambda)}}.$$

Similarly to the developments in Section 3, we obtain

$$\mathbf{E}[C_k(t,0)] = \mathcal{L}^{-1}\left(\frac{\lambda/2}{s\sqrt{s(s+\lambda)}-s^2}\right) =$$

$$= \frac{1}{2} + \frac{1}{2}\left((1+\lambda t)\,I_0(\lambda t) + \lambda t I_1(\lambda t)\right)e^{-\lambda t/2}, \quad t \ge 0. \tag{5}$$

***Remark* 1.** We should note that $\mathbf{E}[C_k(t,0)]$ does not depend on the particle's velocity v. It follows from Eq. (5) that

$$\lim_{\lambda\to\infty}\frac{\mathbf{E}[C_k(t,0)]}{\sqrt{\lambda}} = \sqrt{\frac{2}{\pi}}\sqrt{t}.$$

In Kac's condition we have $\lambda = \varepsilon^{-2}$, $v = \varepsilon^{-1}$, then

$$\mathbf{E}[C_k(t,0)] \sim \sqrt{\frac{2}{\pi}}\sqrt{t}\,v \quad \text{as} \quad \varepsilon \to 0.$$

Let $\{\nu(t),\, t \ge 0\}$ be a renewal process such that $\nu(t) = \max\{m \ge 0\colon \tau_m \le t\}$, where $\tau_m = \sum_{k=1}^m \theta_k$, $\tau_0 = 0$ and $\theta_k \ge 0$, $k = 1, 2, \ldots$, are nonnegative iid random variables denoting the interarrival times. We assume that these random variables have a cdf $G(t)$ and that there exists the pdf $g(t) = \frac{d}{dt}G(t)$. In most of this paper, we have considered exponentially distributed interrenewal times, i.e., $g(t) = \lambda e^{-\lambda t} I_{\{t \geq 0\}}$.

We will study the random motion of a particle that starts from the coordinate origin $\mathbf{0} = (0, 0, \ldots, 0)$ of the space $\mathbb{R}^n$, at time $t = 0$, and continues its motion with a velocity $v > 0$ along the direction $\boldsymbol{\eta}_0^{(n)}$, where $\boldsymbol{\eta}_i^{(n)} = (x_{i1}, x_{i2}, \ldots, x_{in}) = (x_1, x_2, \ldots, x_n)$, $i = 0, 1, 2, \ldots$, are iid random n-dimensional vectors uniformly distributed on the unit sphere $\Omega_1^{n-1} = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \ldots + x_n^2 = 1\}$.

At instant τ_1 the particle changes its direction to $\boldsymbol{\eta}_1^{(n)} = (x_{11}, x_{12}, \ldots, x_{1n})$, and the particle continues its motion with a velocity $v > 0$ along the direction of $\boldsymbol{\eta}_1^{(n)}$. Then, at instant τ_2 the particle changes its direction to $\boldsymbol{\eta}_2^{(n)} = (x_{21}, x_{22}, \ldots, x_{2n})$, and continues its motion with a velocity v along the direction of $\boldsymbol{\eta}_2^{(n)}$, and so on.

Denote by $\mathbf{X}^{(n)}(t)$, $t \geq 0$, the particle position at time t. We have that

$$\mathbf{X}^{(n)}(t) = v \sum_{j=1}^{\nu(t)} \boldsymbol{\eta}_{j-1}^{(n)} (\tau_j - \tau_{j-1}) + v\, \boldsymbol{\eta}_{\nu(t)}^{(n)} (t - \tau_{\nu(t)}). \tag{6}$$

Basically, Eq. (6) determines the random evolution in the semi-Markov (or renewal) medium $\nu(t)$.

Thus, $\nu(t)$ denotes the number of velocity alternations occurred in the interval $(0, t)$.

The probabilistic properties of the random vector $\mathbf{X}^{(n)}(t)$ are completely determined by those of its projection $X^{(n)}(t) = v \sum_{j=1}^{m} \eta_{j-1}^{(n)} (\tau_j - \tau_{j-1}) + v\, \eta_{\nu(t)}^{(n)} (t - \tau_{\nu(t)})$ on a fixed line, where $\eta_j^{(n)}$ is the projection of $\boldsymbol{\eta}_j^{(n)}$ on the line.

Indeed, let us consider the cdf $F_X(y) = P\left(X^{(n)}(t) \leq y\right)$. Then, the characteristic function (Fourier transform) $H(t, \alpha) = H(t)$ of $\mathbf{X}^{(n)}(t)$, where $\alpha = \|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2}$, is given by

$$H(t) = \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{X}^{(n)}(t)\right)\right\}\right] = \mathbf{E}\left[\exp\left\{i\,\|\boldsymbol{\alpha}\|\left(\mathbf{e}, \mathbf{X}^{(n)}(t)\right)\right\}\right] =$$

$$= \mathbf{E}\left[\exp\left\{i\,\alpha X^{(n)}(t)\right\}\right] = \int_0^\infty \exp\{i\,\alpha y\}\, dF_X(y),$$

where $X^{(n)}(t)$ is the projection of $\mathbf{X}^{(n)}(t)$ onto the unit vector $\mathbf{e}$ and it has a cdf $F_X(y)$.

Let us denote by $f_{\eta^{(n)}}(x)$ the pdf of the projection $\eta_j^{(n)}$ of the vector $\boldsymbol{\eta}_j^{(n)}$ onto a fixed line. It is shown in [9] that $f_{\eta^{(n)}}(x)$ is of the following form:

$$f_{\eta^{(n)}}(x) = \begin{cases} \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\dfrac{n-1}{2}\right)} (1 - x^2)^{(n-3)/2}, & \text{if} \quad x \in [-1, 1], \\ 0, & \text{if} \quad x \notin [-1, 1]. \end{cases}$$

Hence, it is not hard to verify that the cdf $G_n(t) = P[v\eta_i^{(n)}\theta_i \leq t]$ is of the form

$$G_n(t) = \begin{cases} \dfrac{1}{2} + \dfrac{\Gamma\left(\dfrac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\dfrac{n-1}{2}\right)} \displaystyle\int_0^1 \lambda e^{-\frac{\lambda t}{vx}} (1-x^2)^{(n-3)/2} dx, & \text{if } t \geq 0, \\ \dfrac{1}{2} - \dfrac{\Gamma\left(\dfrac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\dfrac{n-1}{2}\right)} \displaystyle\int_0^1 \lambda e^{\frac{\lambda t}{vx}} (1-x^2)^{(n-3)/2} dx, & \text{if } t < 0. \end{cases}$$

Denote by $C_k(t,0)$ the number of crossing the hyperplane $\mathcal{H} = \{x_1 = c = \text{constant}\}$ of the space $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n)\}$, $x_i \in \mathbb{R}$, under the condition that the stochastic process starts from the hyperplane $\mathcal{H}$. The number $C_k(t,0)$ is also equal to the number of crossing of the level $x_1 = c$ by the projection $X^{(n)}(t)$ of $\mathbf{X}^{(n)}(t)$ on the line $\ell = \{(t, 0, \ldots, 0)\}$ such that $t \in \mathbb{R}$.

In accordance with Remark 1 the mean value $\mathbf{E}[C_k(t,0)]$ does not depend on the particle's velocity v.

Therefore,

$$\mathbf{E}[C_k(t,0)] = \frac{1}{2} + \frac{1}{2}\left((1+\lambda t) I_0(\lambda t) + \lambda t\, I_1(\lambda t)\right) e^{-\frac{\lambda}{2}t}.$$

Under Kac's condition we have $\lambda = \varepsilon^{-2}$, $v = \varepsilon^{-1}$, then

$$\mathbf{E}[C_k(t,0)] \sim \sqrt{\frac{2}{\pi}}\sqrt{t}\, v \qquad \text{as} \qquad \varepsilon \to 0.$$

It is well-known that under Kac's condition $\mathbf{X}^{(n)}(t)$ weakly converges to an n-dimensional Wiener process $\mathbf{W}(t)$ [10].

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